

EXPLICIT BOUNDS FOR COMPOSITE LACUNARY POLYNOMIALS

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ABSTRACT. Let $f, g, h \in \mathbb{C}[x]$ be non-constant complex polynomials satisfying $f(x) = g(h(x))$ and let f be lacunary in the sense that it has at most l non-constant terms. Zannier proved in [9] that there exists a function $B_1(l)$ on \mathbb{N} , depending only on l and with the property that $h(x)$ can be written as the ratio of two polynomials having each at most $B_1(l)$ terms. Here, we give explicit estimates for this function or, more precisely, we prove that one may take for instance

$$B_1(l) = (4l)^{(2l)^{(3l)^{l+1}}}.$$

Moreover, in the case $l = 2$, a better result is obtained using the same strategy.

1. INTRODUCTION

Let $f, g, h \in \mathbb{C}[x]$ and $f = g \circ h$ be a lacunary polynomial with l non-constant terms, i.e. f is of the form $f(x) = a_0 + a_1x^{n_1} + \dots + a_lx^{n_l}$. Note that only the number of terms is viewed as fixed, while the coefficients and the degrees may vary. In [9], it was shown by Zannier that there exists a function $B_1(l)$ such that $h(x)$ can be written as the ratio of two polynomials in $\mathbb{C}[x]$ both having no more than $B_1(l)$ terms. In order to give explicit estimates for $B_1(l)$, we are following the strategy of [9, Prop. 2]. Therefore, we will recall those parts of Zannier's proof, which are relevant to our arguments, in the very beginning of the paper. We prove

Theorem 1. *Let $f, g, h \in \mathbb{C}[x]$ be non-constant complex polynomials such that $f(x) = g(h(x))$ has at most l non-constant terms. Then $h(x)$ can be written as the ratio of two polynomials in $\mathbb{C}[x]$ having each at most $B_1(l)$ terms, where*

$$B_1(l) = (4l)^{(2l)^{(3l)^{l+1}}} \text{ for } l \geq 1.$$

There are quite different notions of *lacunarity* (lacunary polynomials are also sometimes called *sparse*). Here, we deal with the situation that the number of terms of a given polynomial is fixed. It was conjectured by Erdős that if g is a complex non-constant polynomial with the property that $g(x)^2$ has at most l terms, then $g(x)$ has also boundedly many terms and their number depends only on l . In [7], Schinzel proved a generalized version of Erdős's

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conjecture, namely the statement not only for $g(x)^2$ but for $g(x)^d$, $d \in \mathbb{N}$. He also extended the conjecture to compositions $f(x) = g(h(x))$, claiming that if f has l terms, then $h(x)$ has at most $B(l)$ terms for some function B on \mathbb{N} . Zannier gave a proof for this (actually in a stronger version), wherein he showed in a first step the existence of a function B_1 such that, under the given assumptions, $h(x)$ can be written as a rational function with at most $B_1(l)$ terms in both the numerator and the denominator [9]. Using this result, he proved the stated claim for representations as polynomials¹ and moreover, he gave a complete description of the general decomposition $f(x) = g(h(x))$ of a given polynomial $f(x)$ with l terms. Also, for an outer composition factor g in $f(x) = g(h(x))$, Zannier gave suitable bounds for the degree of g depending only on the number of terms of f [8]. Note that in both, the polynomial case and the case of a rational function, the special shapes $h(x) = ax^m + b$ and $h(x) = ax^m + bx^{-m} + c$, respectively, must be taken into account. This follows easily from the following observation. Let $h(x) = ax^m + b$ and $g(x) = g_1(x - b)$. Then $f(x) = g(h(x)) = g_1(ax^m)$ has at most l non-constant terms, whereas the degree of g can be arbitrary high. Similarly, in the case of rational functions, one can take for instance $h(x) = x + x^{-1}$ and $g(x) = T_n(x)$, the n -th Chebyshev-polynomial, to get a contradiction with the given statements. However, the results were later extended first to Laurent-polynomials [10] and then to rational functions [5]. Recently, in [3] Fuchs, Mantova and Zannier achieved a final result for completely general algebraic equations $f(x, g(x)) = 0$. Here, $f(x, y)$ is assumed to be monic and of given degree in y and with boundedly many terms in x . As pointed out in [4], there are also other forms of lacunarity. Here, Fuchs and Pethő considered rational functions having only a given number of zeros and poles and they again studied their decomposability. This can be seen as a multiplicative analogue to the above mentioned problem. Based on their results, by computational experiments Pethő and Tengeley studied the decomposability of rational functions having at most four zeros and poles and they provided parametrizations of all possible solutions and the appropriate varieties in this case [6].

The present paper is organized as follows. In the very beginning, we give the main results and parts of Zannier's proof, which are crucial for our deductions. Based on this, in Section 3 we prove our result, giving an explicit bound for the function $B_1(l)$ in Zannier's Proposition. This bound happens to be triple-exponential. The reason for this is the following. The argument in our proof will use an estimate for the ratio n_l/n_1 , which appears itself in an exponent. Such an estimate can be found through an recursive procedure, pointed out by Zannier. The bound we obtain through this method will be double-exponential, which in the end leads to the received order. We do not

¹Examples as $h(x) = (x^n - 1)/(x - 1) = x^{n-1} + x^{n-2} + \dots + x + 1$ show that, written as a polynomial, $h(x)$ can have substantially more terms than in a representation as a rational function.

know whether that can be improved in general or how far away we are from a “good bound”. Anyhow, we believe that this bound is probably far from the truth. At least for the cases $l = 1$ and $l = 2$ we have the smaller estimates 2 and 262 146, respectively. For the latter bound, the corresponding statement and its proof is given in the last part of Section 3.

Finally, we mention that independently of us Dona in [1] also gave an explicit bound for $B_1(l)$. His deductions are based on Zannier’s proof too. Therefore it is not surprising that qualitatively his bound is of the same shape (i.e. also triple-exponential) although it is quantitatively better than ours. Nevertheless, and also for the fact that our result is already mentioned in [2], it appears to us that the result is still worth to be found in the literature.

2. RESULTS FROM ZANNIER’S PROOF

Let $\deg f = m = n_l$ and $\deg g = d$, so that $\deg h = n_l/d$. From [8, Thm. 1] it follows that $d \leq 2l(l-1)$. Set $y = 1/x$ and $\tilde{h}(y) = x^{-n_l/d}h(x) = y^{n_l/d}h(1/y)$. Moreover, write $f(x) = ax^m(1 + b_1y^{n_1} + \dots + b_ly^{n_l})$, where $0 =: n_0 < n_1 < \dots < n_l$. We may assume that f has exactly l non-constant terms, i.e. $ab_1 \dots b_l \neq 0$. Instead of $h(x)$, we are considering $\tilde{h}(y)$, which in fact has the same number of terms as $h(x)$. Also, its degree is bounded by $\deg \tilde{h} \leq n_l/d = \deg h$.

For an integer p with $1 \leq p \leq l-1$, write $\delta_p(x) = 1 + b_1x^{n_1} + \dots + b_px^{n_p}$. Zannier then deduces that, in $\mathbb{C}[[y]]$, \tilde{h} is of the form $\tilde{h}(y) = t_1 + t_2 + \dots + t_L + O(y^{2n_l})$, where L is an integer such that $L \leq (2d+1)(2n_l/n_{p+1}+1)^l$ and the t_1, \dots, t_L are all of the shape

$$c\delta_p(y)^{s/d-k}y^{h_1n_{p+1}+\dots+h_{l-p}n_p+(1-s)m/d}, \quad (1)$$

for varying $h_1, \dots, h_{l-p} \in \mathbb{N}$, suitable constants $c = c(h_1, \dots, h_{l-p}, s)$ and $s \in \{1, 0, -1, \dots, 1-2d\}$ (where in fact it turns out that only $s = 0, 1$ may contribute to \tilde{h}).

In the case that $t_1, \dots, t_L, \tilde{h}(y)$ are linearly independent over \mathbb{C} , it then follows that

$$n_l \leq 16^{l+1}d^3(n_l/n_{p+1})^{2l}(1+n_p), \quad (2)$$

and furthermore, using $1+n_p \leq 2n_p$,

$$(n_l/n_p) \leq 2 \cdot 16^{l+1}d^3(n_l/n_{p+1})^{2l}. \quad (3)$$

If on the other hand $t_1, \dots, t_L, \tilde{h}(y)$ are linearly dependent over \mathbb{C} , it follows inductively that $\tilde{h}(y)$ can be written as the ratio of two polynomials in $\mathbb{C}[y]$ having each at most $B_2(l, n_l/n_{p+1})$ terms, where $B_2(l, u)$ is a suitable function which may be estimated in terms of $B_1(l-1)$ and of $u \geq 0$.

Now, one distinguishes between those two cases for each $p = l-1, l-2, \dots$. Suppose that for $p = l-1, l-2, \dots, l-r$ always the first situation occurs.

In that case we can recursively determine upper bounds for the quotients n_l/n_p , since for $p = l - 1$ the initial condition $n_l/n_{p+1} = 1$ holds.

The proof then argues via backwards induction on $p = l - 1, l - 2, \dots$. We distinguish between two possible cases. The first one is the case that $t_1, t_2, \dots, t_L, \tilde{h}(y)$ are linearly independent for all $p = l - 1, l - 2, \dots, 1$. Here, one can use (3) and (2) to get an estimate for n_l , which in fact also gives an estimate for the number of terms of $\tilde{h}(y)$ (and hence of $h(x)$) as a polynomial, since the number of terms of \tilde{h} is bounded by its degree $n_l/d \leq n_l$.

In the other case, namely if the $t_1, \dots, t_L, \tilde{h}(y)$ are linearly dependent over \mathbb{C} for at least one $p \in \{1, \dots, l - 1\}$, let p_0 denote the last p for which this case occurs, so we may assume that we have linear independency for $p = l - 1, l - 2, \dots, p_0 + 1$ and linear dependency for $p = p_0$. Here, Zannier concludes that $\tilde{h}(y)$ is the sum of at most $L = L(p_0)$ terms of the form (1), where $k \leq 2l \cdot n_l/n_{p_0+1} =: \tilde{k}$, $s \in \{1, 0, -1, \dots, 1 - 2d\}$. Now, one can use the linear independency in the cases $p > p_0$ to estimate the quotient n_l/n_{p_0+1} , which occurs in the estimates for L and \tilde{k} .

3. PROOF OF THEOREM 1 AND THE CASE $l = 2$

In order to prove Proposition 1, we start with the following lemma.

Lemma 1. *Let $f \in \mathbb{C}[x]$ be of the form $f(x) = a_0 + a_1x^{n_1} + \dots + a_lx^{n_l}$, $0 < n_1 < \dots < n_l$, and let for every integer p , $1 \leq p \leq l - 1$, $S = S(p) = \{t_1, \dots, t_L, \tilde{h}(y)\}$ be the set described in Section 2. If for each $p = l - 1, \dots, l - r$ the set S is linearly independent over \mathbb{C} , it holds*

$$\frac{n_l}{n_{l-r}} \leq (16^{l+2}l^6)^{(3l)^{r-1}}.$$

Proof. We set $\lambda = 2 \cdot 16^{l+1}d^3$. By [8, Thm. 1], we have $d \leq 2l(l - 1)$. Applying (3), we obtain iteratively

$$\begin{aligned} \frac{n_l}{n_{l-1}} &\leq 2 \cdot 16^{l+1} \cdot d^3 =: \lambda, \\ \frac{n_l}{n_{l-2}} &\leq 2 \cdot 16^{l+1} \cdot d^3 (2 \cdot 16^{l+1} \cdot d^3)^{2l} = \lambda^{2l+1}, \\ \frac{n_l}{n_{l-3}} &\leq \lambda (\lambda^{2l+1})^{2l} = \lambda^{1+2l(1+2l)}, \\ &\vdots \\ \frac{n_l}{n_{l-r}} &\leq \lambda^{1+2l(1+2l(1+2l(\dots)))} \leq \lambda^{(3l)^{r-1}}. \end{aligned}$$

As $\lambda = 2 \cdot 16^{l+1}d^3 \leq 2 \cdot 16^{l+1}(2l(l - 1))^3 \leq 16^{l+2}l^6$, in the case that for $p = l - 1, l - 2, \dots, l - r$ the set S is always linearly independent over \mathbb{C} , we obtain the claimed result. \square

Proof of Proposition 1. Following the proof of [9, Prop. 2], we argue by induction on l . As pointed out by Zannier, for $l = 1$ we may take $B_1(1) \geq 2$. Now, for the rest of the proof let us assume B_1 has been suitably defined on $\{1, 2, \dots, l-1\}$.

For $p = l-1, l-2, \dots, 0$ and $\delta_p(y) = 1 + b_1 y^{n_1} + \dots + b_p y^{n_p}$, we can write in $\mathbb{C}[[y]]$

$$\tilde{h}(y) = t_1 + t_2 + \dots + t_L + O(y^{2n_l}), \quad (4)$$

where the t_i , $1 \leq i \leq L$, are terms of the form (1) and L is an integer which, as we know from the proof of [9, Prop. 2], can be bounded by $L \leq (2d+1)(2(n_l/n_{p+1})+1)^l$. Using the fact that $d \leq 2l(l-1)$, we can make the further rough estimate

$$L \leq (2(2l(l-1)) + 1)(3 \frac{n_l}{n_{p+1}})^l < 4l^2(3 \frac{n_l}{n_{p+1}})^l. \quad (5)$$

We now consider the set $S = \{t_1, \dots, t_L, \tilde{h}(y)\}$ and we distinguish between two possible cases, namely that $S = S(p)$ is linearly independent over \mathbb{C} for every $p = l-1, l-2, \dots, 1$ or that there is an integer p_0 , $1 \leq p_0 \leq l-1$, such that we have linear dependency, i.e. $S = S(p)$ is linearly independent for $p = l-1, l-2, \dots, p_0+1$ and $S(p_0)$ is linearly dependent over \mathbb{C} .

Case 1. In the case of linear independency for every $p = l-1, l-2, \dots, 1$, by Lemma 1, we get the estimate $n_l/n_1 \leq (16^{l+2}l^6)^{(3l)^{l-2}}$. Now, we may apply (2) for $p = 0$ to get

$$\begin{aligned} n_l &\leq 16^{l+1}d^3(16^{l+2}l^6)^{(2l)(3l)^{l-2}} \\ &\leq 16^{l+1}(2l^2)^3(16^{l+2}l^6)^{(2l)(3l)^{l-2}} \\ &< (16^{l+2}l^6)^{(2l)(3l)^{l-2}+1} \\ &< (16^{l+2}l^6)^{(3l)^{l-1}}. \end{aligned}$$

This in fact also gives an estimate for the number of terms of $\tilde{h}(y)$ written as a polynomial (and hence of $h(x)$), since the number of terms of \tilde{h} is bounded by its degree $n_l/d \leq n_l$.

Case 2. Let us now consider the second case, where we have linear dependency for some $p = p_0$. Since we assume $S = S(p)$ to be linear independent over \mathbb{C} for each $p > p_0$, we have, again by Lemma 1, that n_l/n_{p_0+1} can be bounded by $(16^{l+2}l^6)^{(3l)^{l-1}}$ as well. Let $e = [K : \mathbb{C}(y)]$, where $K = \mathbb{C}(y, \delta_p(y)^{1/d})$, so $e \in \{1, \dots, d\}$ is the least integer such that $\delta_p(y)^e$ is a d -th power in $\mathbb{C}(y)$. We write $\delta_p(y)^e = \eta_p(y)^d$ for a polynomial $\eta_p \in \mathbb{C}[y]$ to express this fact. From the proof of [9, Prop. 2], we know that $\tilde{h}(y) = \Lambda_0$, where Λ_0 is the sum of at most L terms of the shape

$$c\delta_p(y)^{s/d-k}y^{(1-s)m/d+h_1n_{p+1}+\dots+h_{l-p}n_p}, \quad (6)$$

with $k \leq 2ln_l/n_{p+1} =: \tilde{k}$ and where $s \in \{0, 1\}$ is such that $e|s$.

In order to estimate the number of terms in the requested representation of \tilde{h} , we first look at $\delta_p(y)^{s/d-k}$, which is the important quantity in (6), when it comes to counting terms. First of all, we consider the case that $s = 1$ contributes to $\tilde{h}(y)$. Consequently, $e = 1$ and we have

$$\delta_p(y)^{s/d-k} = \frac{\delta_p(y)^{s/d}}{\delta_p(y)^k} = \frac{\eta_p(y)^{s/e} \delta_p(y)^{\tilde{k}-k}}{\delta_p(y)^{\tilde{k}}} = \frac{\eta_p(y)^s \delta_p(y)^{\tilde{k}-k}}{\delta_p(y)^{\tilde{k}}}. \quad (7)$$

Here, $\eta_p(y)^{d/e} = \delta_p(y)$ is a polynomial with at most $p \leq l-1$ non-constant terms, so by the induction hypothesis $\eta_p(y)$ can be expressed as a rational function with at most $B_1(l-1)$ terms in both the numerator and the denominator. That is,

$$\eta_p(y) = \frac{\eta_{p,1}(y)}{\eta_{p,2}(y)},$$

where $\eta_{p,1}, \eta_{p,2}$ are complex polynomials with at most $B_1(l-1)$ terms. Now, let us again consider $\tilde{h}(y) = \Lambda_0$. With (7) in mind, after reducing all of the L terms of the shape (6) to the common denominator $\eta_{2,p}(y) \delta_p(y)^{\tilde{k}}$, we can make the rough estimate $(p+1)^{\tilde{k}} B_1(l-1)$ for the number of terms in the denominator and $(p+1)^{\tilde{k}} B_1(l-1)L$ for the number of terms in the numerator. The case that $s = 0$ holds in every term can be treated analogously and leads to the smaller bound $(p+1)^{\tilde{k}}$ and $(p+1)^{\tilde{k}}L$, respectively.

Since we are looking for a function that bounds both the number of terms in the numerator and the denominator, it now obviously suffices to give an estimate for $(p+1)^{\tilde{k}} B_1(l-1)L$, depending only on l . From now on, we may assume $l \geq 2$, since for $l = 1$ it already has been shown in [9, Prop. 2] that we may take $B_1(1) \geq 2$. Using that $p+1 \leq l$, $\tilde{k} = 2l n_l / n_{p+1}$, $L \leq 4l^2 3^l (n_l / n_{p+1})^l$ by (5) and $n_l / n_{p+1} \leq (16^{l+2} l^6)^{(3l)^{l-2}}$ by Lemma 1, we further get

$$B_1(l-1)(p+1)^{\tilde{k}}L \leq B_1(l-1)l^{2l(16^{l+2}l^6)^{(3l)^{l-2}}} \cdot 4l^2 3^l (16^{l+2}l^6)^{l(3l)^{l-2}} \quad (8)$$

$$\begin{aligned} &= B_1(l-1)l^{2l(16^{l+2}l^6)^{(3l)^{l-2}}} l^{2l6l(3l)^{l-2}} \cdot 4 \cdot 3^l \cdot 16^{l(l+2)(3l)^{l-2}} \\ &\leq B_1(l-1)l^{2l(16^{l+2}l^6)^{(3l)^{l-2}} + 2+2(3l)^{l-1}} 4^{1+l+2l(l+2)(3l)^{l-2}}. \end{aligned} \quad (9)$$

We continue to estimate the exponent of l . Therefore, we use that

$$2 + 2(3l)^{l-1} < l(3l)^{l-1} + 2l(3l)^{l-1} = (3l)^l < (4l)^l = 2^{2l}l^l. \quad (10)$$

We have

$$\begin{aligned} 2l(16^{l+2}l^6)^{(3l)^{l-2}} + 2 + 2(3l)^{l-1} &< 2^{4(l+2)(3l)^{l-2}+1} l^{6(3l)^{l-2}+1} + 2^{2l}l^l \\ &< 2^{4(l+2)(3l)^{l-2}+1} l^{6(3l)^{l-2}+1} \cdot 2 \\ &= 2^{4(l+2)(3l)^{l-2}+2} l^{6(3l)^{l-2}+1} \\ &< (2l)^{(3l)^l}. \end{aligned} \quad (11)$$

As for l , we also estimate the exponent of 4 in (9):

$$1 + l + 2l(l+2)(3l)^{l-2} = (3l)^{l-2} \left(\frac{l+1}{(3l)^{l-2}} + 2l^2 + 4l \right) < (3l)^{l-2} (3l)^2 = (3l)^l. \quad (12)$$

Using (11) and (12), we continue and get in (9)

$$B_1(l-1)(p+1)^{\tilde{k}} L \leq B_1(l-1) l^{(2l)(3l)^l} 4^{(3l)^l}.$$

It has been shown in [9, Prop. 2] that for $l = 1$ we may take $B_1(1) \geq 2$. Hence, if we set $\tilde{B}(1) = 2$ and $\tilde{B}(l) = \tilde{B}(l-1) l^{(2l)(3l)^l} 4^{(3l)^l}$ for $l \geq 2$, it suffices to take for $B_1(l)$ any function satisfying $B_1(l) \geq \tilde{B}(l)$ and, keeping the result from *Case 1* in mind, $B_1(l) \geq (16^{l+2} l^6)^{(3l)^{l-1}}$. From the definition of $\tilde{B}(l)$ we get

$$\tilde{B}(l) = 2 \prod_{k=2}^l k^{(2k)^{(3k)^k}} 4^{(3k)^k}.$$

Since $(2k)^{(3k)^k} = 2^{3^k k^k} k^{(3k)^k} = (2^{3^k})^{k^k} k^{(3k)^k} > 3^k k^k$ holds for $k \geq 2$, it follows that $k^{(2k)^{(3k)^k}} 4^{(3k)^k+1} \leq (4k)^{(2k)^{(3k)^k}}$, so we have

$$\begin{aligned} \tilde{B}(l) &< \prod_{k=2}^l k^{(2k)^{(3k)^k}} 4^{(3k)^k+1} < \prod_{k=2}^l (4k)^{(2k)^{(3k)^k}} < \left[(4l)^{(2l)(3l)^l} \right]^l = \\ &= (4l)^{l(2l)(3l)^l} < (4l)^{(2l)(3l)^{l+1}} < (4l)^{(2l)(3l)^{l+1}}. \end{aligned}$$

Finally, we have to check that the obtained estimate also holds in *Case 1*. Therefore, we need $(16^{l+2} l^6)^{(3l)^{l-1}} \leq (4l)^{(2l)(3l)^{l+1}}$ for $l \geq 2$. We have

$$(16^{l+2} l^6)^{(3l)^{l-1}} = (4^{2l+4} l^6)^{(3l)^{l-1}} < (4l)^{(2l)(2(3l)^{l-1})}. \quad (13)$$

Hence, we need $(2l)^{(3l)^{l+1}-1} > 2(3l)^{l-1}$. Since $(2l)^{x-1} > 2^{x-1} > 2x$ holds for every $x \geq 5$, for $l \geq 2$ we obtain

$$(2l)^{(3l)^{l+1}-1} > (2l)^{(3l)^{l-1}-1} > 2(3l)^{l-1}.$$

So, we finally have shown that the conditions required in Proposition 1 are fulfilled if we set

$$B_1(l) = (4l)^{(2l)(3l)^{l+1}}.$$

□

Using the same arguments, it is also possible to obtain a better bound for $l = 2$, since in this case we are able to keep the estimates during the proof essentially smaller. Eventually, we get the following.

Proposition 1. *Let $f, g, h \in \mathbb{C}[x]$ be non-constant complex polynomials such that $f(x) = g(h(x))$ has at most 2 non-constant terms. Then $h(x)$ may be written as the ratio of two polynomials in $\mathbb{C}[x]$ having each at most 262146 terms.*

Proof. Assume that f is a polynomial in $\mathbb{C}[x]$ with at most two non-constant terms, i.e. $f(x) = a_0 + a_1x^{n_1} + a_2x^{n_2}$, $0 < n_1 < n_2$. We keep the notations from above. If $\deg g = d = 1$, the number of terms of $g(h(x))$ and of $h(x)$ may only deviate by one (namely the constant term), hence in this case $h(x)$ is a polynomial that has not more than 3 terms. So in the following we assume that $\deg g \geq 2$. As it has been shown in [9], for certain $\gamma_{-1}, \gamma_0, \gamma_1, \dots \in \mathbb{C}$ we have in $\mathbb{C}[[y]]$

$$\tilde{h}(y) = \gamma_{-1}\tilde{f}(y)^{1/d} + \gamma_0y^{m/d} + \gamma_1y^{2m/d}\tilde{f}(y)^{-1/d} + \gamma_2y^{3m/d}\tilde{f}(y)^{-2/d} + \dots,$$

where the roots $\tilde{f}(y)^{s/d}$ are of the form

$$\tilde{f}(y)^{s/d} = \sum_{(h_1, h_2) \in \mathbb{N}_0^2} c_{s,d,(h_1, h_2)} b_1^{h_1} b_2^{h_2} y^{h_1 n_1 + h_2 n_2},$$

for certain universal coefficients $c_{s,d,(h_1, h_2)}$. Since $\tilde{h}(y)$ is a polynomial with $\deg \tilde{h} \leq m/d = n_2/d$, it follows that $\tilde{h}(y)$ is the sum of $\gamma_0y^{m/d}$ and the terms of $\gamma_{-1}\tilde{f}(y)^{1/d}$, for which $h_1 n_1 + h_2 n_2 \leq n_2/d$ holds. Since $d \leq 2l(l-1)$, we have $2 \leq d \leq 4$, so the only pairs $(h_1, h_2) \in \mathbb{N}_0^2$ possibly satisfying $h_1 n_1 + h_2 n_2 \leq n_2/d$ are $(0, 0), (1, 0), \dots, (\lfloor n_2/(2n_1) \rfloor, 0)$, which gives all in all at most $\lfloor n_2/(2n_1) \rfloor + 2$ terms.

We now again consider the two cases that the set $S = S(p) = \{t_1, \dots, t_L, \tilde{h}(y)\}$ is linearly dependent or linearly independent over \mathbb{C} , respectively. In general, we have to distinguish between those cases for each $p = l-1, l-2, \dots$, but since $l = 2$, $p = 1$ is the only remaining situation we have to look at.

Case 1. From the previous proof we know that in the case of linear dependency the number of terms can be estimated by $(p+1)^{\tilde{k}} \tilde{B}_1(1)L$, where $L \leq (2d+1)(2n_l/n_{p+1}+1)^l$. Therefore, we get $L \leq (2 \cdot 4 + 1)(2n_2/n_2 + 1)^2 = 81$. We also have $\tilde{k} = 2ln_l/n_{p+1}$, which gives $\tilde{k} = 4$ as well. So, with $\tilde{B}_1(1) = 2$ we get the estimate

$$(p+1)^{\tilde{k}} \cdot \tilde{B}_1(1) \cdot L \leq 2^4 \cdot 2 \cdot 81 = 2^5 \cdot 3^4 = 2592.$$

Case 2. In the case of linear independency over \mathbb{C} , from

$$n_2 \leq (16d)^3(1+n_1) \leq (16 \cdot 4)^3 \cdot 2n_1 = 2^{19}n_1,$$

we get $n_2/n_1 \leq 2^{19}$ as an estimate for the quotient of the exponents. So, by the argument we gave at the beginning of the proof, we find the estimate $2^{18} + 2 = 262146$.

Finally, we find that we can take this as an upper bound in every case. So we can conclude that, for a polynomial which has at most two non-constant

terms, there exists a representation as a rational function such that both the numerator and the denominator have no more than 262 146 terms. \square

The obtained bound is still not very small. Anyway, compared to the bound $B_1(2) = 2^{3 \cdot 2^{432}}$, where $2^{3 \cdot 2^{432}} > 10^{2^{431}}$ and the exponent 2^{431} already has 130 digits, this still gives a notable improvement.

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